

Chapter 2: Problem Solutions

Discrete Time Processing of Continuous Time Signals

Sampling

■ Problem 2.1.

Problem:

Consider a sinusoidal signal

$$x(t) = 3 \cos(1000\pi t + 0.1\pi)$$

and let us sample it at a frequency $F_s = 2$ kHz.

- Determine an expression for the sampled sequence $x[n] = x(nT_s)$ and determine its Discrete Time Fourier Transform $X(\omega) = \text{DTFT}\{x[n]\}$;
- Determine $X(F) = \text{FT}\{x(t)\}$;
- Recompute $X(\omega)$ from the $X(F)$ and verify that you obtain the same expression as in a).

Solution:

a) $x[n] = x(t) |_{t=nT_s} = 3 \cos(0.5\pi n + 0.1\pi)$. Equivalently, using complex exponentials,

$$x[n] = 1.5 e^{j0.1\pi} e^{j0.5\pi n} + 1.5 e^{-j0.1\pi} e^{-j0.5\pi n}$$

Therefore its DTFT becomes

$$X(\omega) = \text{DTFT}\{x[n]\} = 3\pi e^{j0.1\pi} \delta(\omega - \frac{\pi}{2}) + 3\pi e^{-j0.1\pi} \delta(\omega + \frac{\pi}{2})$$

with $-\pi < \omega < \pi$

b) Since $\text{FT}\{e^{j2\pi F_0 t}\} = \delta(F - F_0)$ then

$$X(F) = 1.5 e^{j0.1\pi} \delta(F - 500) + 1.5 e^{-j0.1\pi} \delta(F + 500)$$

for all F .

c) Recall that $X(\omega) = \text{DTFT}\{x[n]\}$ and $X(F) = \text{FT}\{x(t)\}$ are related as

$$X(\omega) = F_s \sum_{k=-\infty}^{+\infty} X(F - kF_s) \Big|_{F=\omega F_s/2\pi}$$

with F_s the sampling frequency. In this case there is no aliasing, since all frequencies are contained within $F_s/2 = 1$ kHz. Therefore, in the interval $-\pi < \omega < +\pi$ we can write

$$X(\omega) = F_s X(F) \Big|_{F=\omega F_s/2\pi}$$

with $F_s = 2000$ Hz. Substitute for $X(F)$ from part b) to obtain

$$X(\omega) = 2000 (1.5 e^{j0.1\pi} \delta(2000 \frac{\omega}{2\pi} - 500) + 1.5 e^{-j0.1\pi} \delta(2000 \frac{\omega}{2\pi} + 500))$$

Now recall the property of the "delta" function: for any constant $a \neq 0$,

$$\delta(at) = \frac{1}{|a|} \delta\left(\frac{t}{a}\right)$$

Therefore we can write

$$X(\omega) = 3\pi e^{j0.1\pi} \delta\left(\omega - \frac{\pi}{2}\right) + 3\pi e^{-j0.1\pi} \delta\left(\omega + \frac{\pi}{2}\right)$$

same as in b).

■ Problem 2.2.

Problem

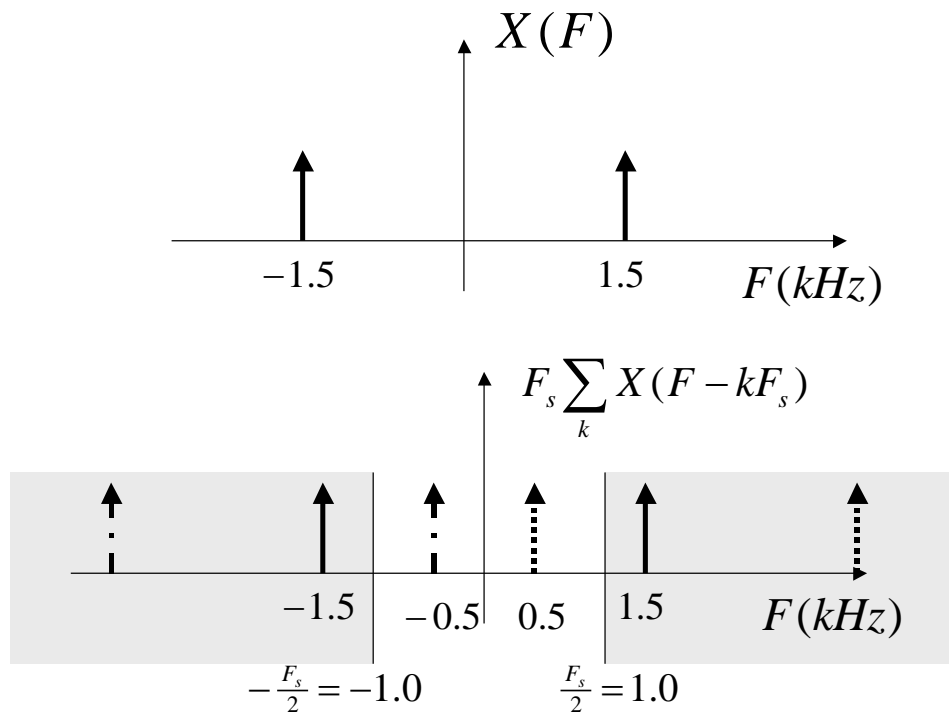
Repeat Problem 1 when the continuous time signal is

$$x(t) = 3 \cos(3000\pi t)$$

Solution

Following the same steps:

a) $x[n] = 3 \cos(1.5\pi n)$. Notice that now we have aliasing, since $F_0 = 1500$ Hz $> \frac{F_s}{2} = 1000$ Hz. Therefore, as shown in the figure below, there is an aliasing at $F_s - F_0 = 2000 - 1500$ Hz = 500 Hz. Therefore after sampling we have the same signal as in Problem 1.1, and everything follows.



■ Problem 2.3.

Problem

For each $X(F) = \text{FT}\{x(t)\}$ shown, determine $X(\omega) = \text{DTFT}\{x[n]\}$, where $x[n] = x(nT_s)$ is the sampled sequence. The Sampling frequency F_s is given for each case.

- $X(F) = \delta(F - 1000)$, $F_s = 3000$ Hz;
- $X(F) = \delta(F - 500) + \delta(F + 500)$, $F_s = 1200$ Hz
- $X(F) = 3 \text{rect}\left(\frac{F}{1000}\right)$, $F_s = 2000$ Hz;
- $X(F) = 3 \text{rect}\left(\frac{F}{1000}\right)$, $F_s = 1000$ Hz;
- $X(F) = \text{rect}\left(\frac{F-3000}{1000}\right) + \text{rect}\left(\frac{F+3000}{1000}\right)$, $F_s = 3000$ Hz;

Solution

For all these problems use the relation

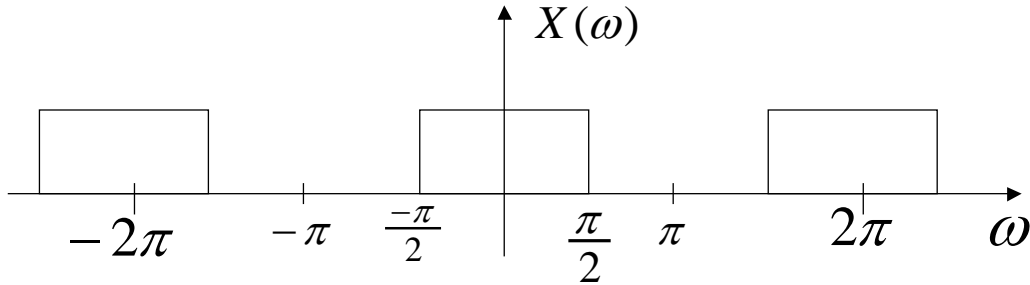
$$X(\omega) = F_s \sum_{k=-\infty}^{+\infty} X(\omega F_s / 2\pi - kF_s)$$

$$\text{a) } X(\omega) = 3000 \sum_{k=-\infty}^{+\infty} \delta\left(\omega \frac{3000}{2\pi} - 1000 - k3000\right) = 2\pi \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi}{3} - k2\pi\right);$$

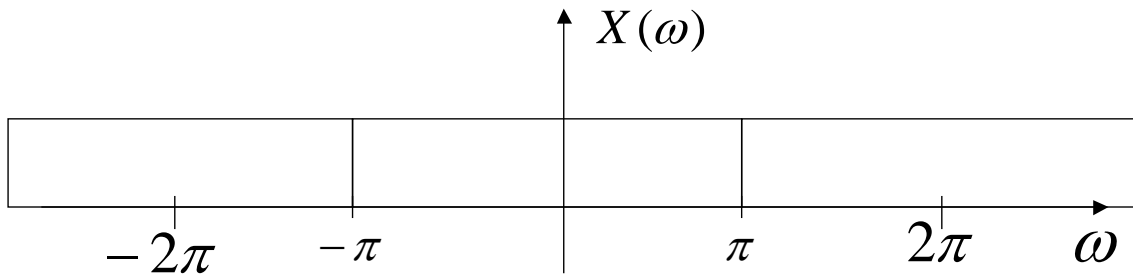
$$\begin{aligned} \text{b) } X(\omega) &= 1200 \sum_{k=-\infty}^{+\infty} \delta\left(\omega \frac{1200}{2\pi} - 500 - k1200\right) + \delta\left(\omega \frac{1200}{2\pi} + 500 - k1200\right) \\ &= 2\pi \sum_{k=-\infty}^{+\infty} \delta(\omega - \omega_0 - k2\pi) + \delta(\omega + \omega_0 - k2\pi) \end{aligned}$$

$$\omega_0 = 2\pi \times 500 / 1200 = \pi / 1.2;$$

$$\text{c) } X(\omega) = 2000 \times 3 \sum_{k=-\infty}^{+\infty} \text{rect}\left(\frac{\omega 2000/2\pi}{1000} - k \frac{2000}{1000}\right) = 6000 \sum_{k=-\infty}^{+\infty} \text{rect}\left(\frac{\omega - k2\pi}{\pi}\right) \text{ shown below.}$$

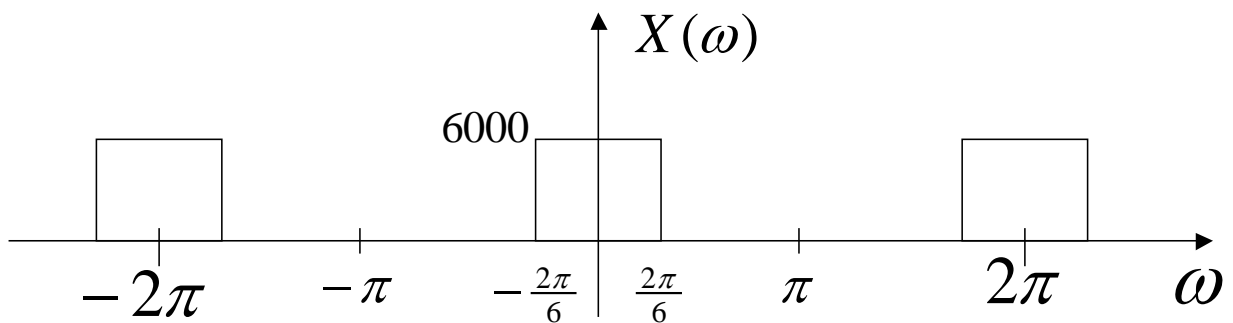


$$\text{d) } X(\omega) = 1000 \times 3 \sum_{k=-\infty}^{+\infty} \text{rect}\left(\frac{\omega 1000/2\pi}{1000} - k \frac{1000}{1000}\right) = 3000 \sum_{k=-\infty}^{+\infty} \text{rect}\left(\frac{\omega - k2\pi}{2\pi}\right) \text{ shown below}$$



$$\begin{aligned} \text{e) } X(\omega) &= 3000 \sum_{k=-\infty}^{+\infty} \text{rect}\left(\frac{\omega 3000/2\pi - 3000}{1000} - k \frac{3000}{1000}\right) + \text{rect}\left(\frac{\omega 3000/2\pi + 3000}{1000} - k \frac{3000}{1000}\right) \\ &= 3000 \sum_{k=-\infty}^{+\infty} \text{rect}\left(\frac{3\omega}{2\pi} - 3 - 3k\right) + \text{rect}\left(\frac{3\omega}{2\pi} + 3 - 3k\right) \\ &= 6000 \sum_{k=-\infty}^{+\infty} \text{rect}\left(\frac{\omega - k2\pi}{2\pi/3}\right) \end{aligned}$$

shown below.



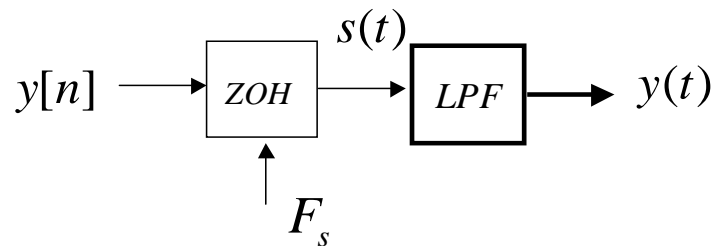
■ Problem 2.4.

Problem

In the system shown, let the sequence be

$$y[n] = 2 \cos (0.3 \pi n + \pi / 4)$$

and the sampling frequency be $F_s = 4$ kHz. Also let the low pass filter be ideal, with bandwidth $F_s / 2$.



- Determine an expression for $S(F) = \text{FT} \{s(t)\}$. Also sketch the frequency spectrum (magnitude only) within the frequency range $-F_s < F < F_s$;
- Determine the output signal $y(t)$.

Solution.

From what we have seen, recall that

$$S(F) = e^{-j\pi F/F_s} \frac{1}{F_s} \text{sinc} \left(\frac{F}{F_s} \right) Y(\omega) \Big|_{\omega=2\pi F/F_s}$$

From $Y(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} e^{j\pi/4} \delta(\omega - 0.3\pi - k2\pi) + e^{-j\pi/4} \delta(\omega + 0.3\pi - k2\pi)$ we obtain

$$\begin{aligned} Y(\omega) \Big|_{\omega=2\pi F/F_s} &= \\ 2\pi \sum_{k=-\infty}^{+\infty} e^{j\pi/4} \delta\left(2\pi \frac{F}{F_s} - 0.3\pi - k2\pi\right) &+ e^{-j\pi/4} \delta\left(2\pi \frac{F}{F_s} + 0.3\pi - k2\pi\right) \\ &= 2\pi \times \frac{F_s}{2\pi} \sum_{k=-\infty}^{+\infty} e^{j\pi/4} \delta(F - 600 - k4000) + \\ &e^{-j\pi/4} \delta\left(2\pi \frac{F}{F_s} + 600 - k4000\right) \end{aligned}$$

and then

$$\begin{aligned} S(F) &= F_s \sum_{k=-\infty}^{+\infty} T_s e^{-j\pi(600+k4000)/4000} \text{sinc}\left(\frac{600+k4000}{4000}\right) e^{j\pi/4} \delta(F - 600 - k4000) + \\ &T_s e^{-j\pi(-600+k4000)/4000} \text{sinc}\left(\frac{-600+k4000}{4000}\right) e^{-j\pi/4} \delta\left(2\pi \frac{F}{F_s} + 600 - k4000\right) \end{aligned}$$

where we used the fact that the ZOH has frequency response $T_s e^{-j\pi F/F_s} \text{sinc}(F/F_s)$.

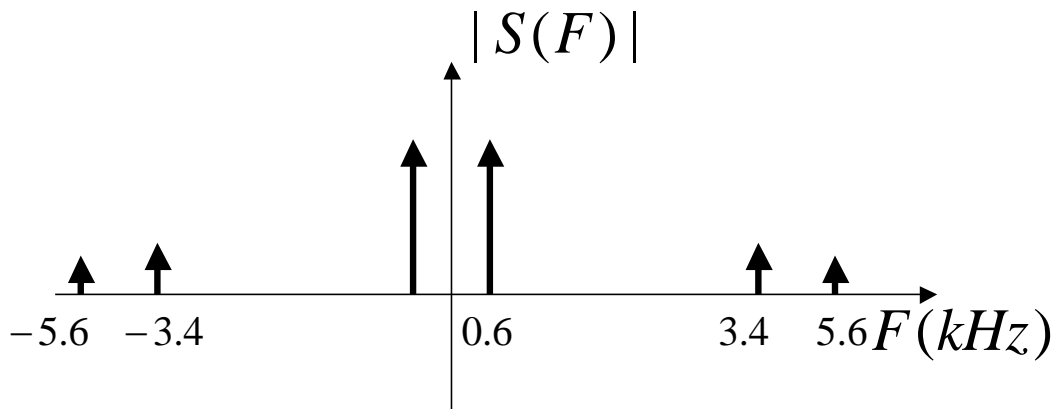
This can be simplified to

$$S(F) = \sum_{k=-\infty}^{+\infty} (-1)^k e^{-j \frac{3\pi}{20}} \operatorname{sinc} \left(\frac{3}{20} + k \right) e^{j\pi/4} \delta(F - 600 - k4000) + (-1)^k e^{-j \frac{3\pi}{20}} \operatorname{sinc} \left(\frac{-3}{20} + k \right) e^{-j\pi/4} \delta(F + 600 - k4000)$$

In the interval $-F_s = -4000 < F < F_s = 4000$ we have only terms corresponding to $k = -1, 0, 1$. The reader can verify that all other frequencies are outside this interval. Therefore, for $-4000 < F < +4000$ we have

$$S(F) = 0.17 e^{-j2 \cdot 827} \delta(F - 3400) + 0.9634 e^{-j0 \cdot 1\pi} \delta(F - 600) + 0.9634 e^{j0 \cdot 1\pi} \delta(F + 600) + 0.17 e^{j2 \cdot 827} \delta(F + 3400)$$

shown below.



b) Since the Low Pass Filter stops all the frequencies above $F_s / 2$ the output signal $y(t)$ has only the frequencies at $F = \pm 600$ Hz, and therefore

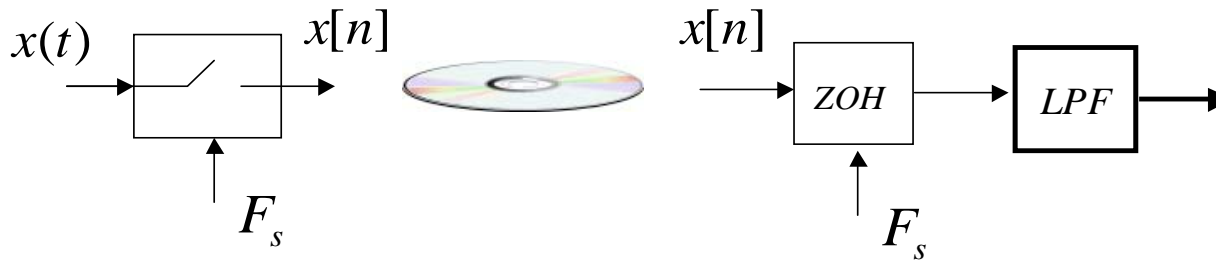
$$y(t) = \text{IFT} \{ 0.9634 e^{j0 \cdot 1\pi} \delta(F + 600) + 0.9634 e^{-j0 \cdot 1\pi} \delta(F - 600) \} = 2 \times 0.9634 \cos(1200 \pi t - 0.1 \pi)$$

■ Problem 2.5.

Problem

We want to digitize and store a signal on a CD, and then reconstruct it at a later time. Let the signal $x(t)$ be

$$x(t) = 2 \cos(500 \pi t) - 3 \sin(1000 \pi t) + \cos(1500 \pi t)$$



and let the sampling frequency be $F_s = 2000$ Hz.

a) Determine the continuous time signal $y(t)$ after the reconstruction.

b) Notice that $y(t)$ is not exactly equal $x(t)$. How could we reconstruct the signal $x(t)$ exactly from its samples $x[n]$?

Solution

a) Recall the formula, *in absence of aliasing*,

$$Y(F) = e^{-j\pi F/F_s} \operatorname{sinc}\left(\frac{F}{F_s}\right) X(F)$$

with $F_s = 2000$ Hz being the sampling frequency. In this case there is no aliasing, since the maximum frequency is 750 Hz smaller than $F_s / 2 = 1000$ Hz. Therefore, each sinusoid at frequency F has magnitude and phase scaled by the above expression. Define

$$G(F) = \frac{e^{-\frac{j\pi F}{2000}} \sin\left(\frac{\pi F}{2000}\right)}{\frac{\pi F}{2000}}$$

which yields

$$G[250] = 0.9745 e^{-j0.392}, \quad G[500] = 0.9003 e^{-j0.785}, \quad G[750] = 0.784 e^{-j1.178}$$

Finally, apply to each sinusoid to obtain.

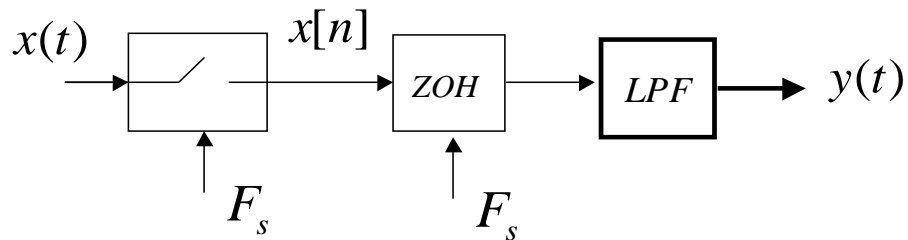
$$y(t) = 2 \times 0.9745 \times \cos(500\pi t - 0.392) - 3 \times 0.9003 \times \sin(1000\pi t - 0.785) + 0.784 \cos(1500\pi t - 1.178)$$

b) In order to compensate for the distortion we can design a filter with frequency response $1/G(F)$, when $-\frac{F_s}{2} < F < \frac{F_s}{2}$. The magnitude would be as follows

■ Problem 2.6.

Problem

In the system shown below, determine the output signal $y(t)$ for each of the following input signals $x(t)$. Assume the sampling frequency $F_s = 5$ kHz and the Low Pass Filter (LPF) to be ideal with bandwidth $F_s / 2$:



- $x(t) = e^{j2000\pi t}$;
- $x(t) = \cos(2000\pi t + 0.15\pi)$;
- $x(t) = 2 \cos(5000\pi t)$;
- $x(t) = 2 \sin(5000\pi t)$;
- $x(t) = \cos(2000\pi t + 0.1\pi) - \cos(5500\pi t)$.

Solution

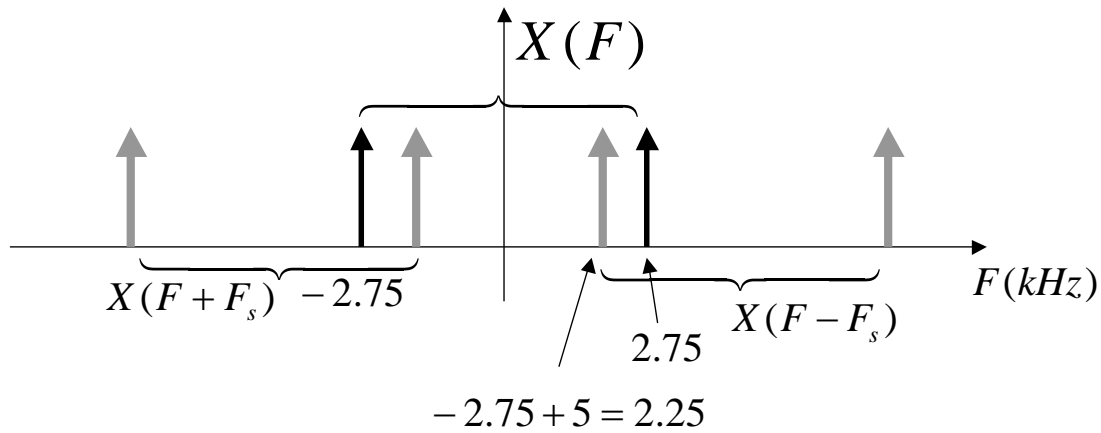
Recall the frequency response, in case of no aliasing, is

$$G(F) = \frac{e^{-j\frac{\pi F}{5000}} \text{Sin}\left(\frac{\pi F}{5000}\right)}{\frac{\pi F}{5000}}$$

with $-2500 < F < 2500$. Then:

- $G(1000) = 0.935 e^{-j0.628}$ and then $y(t) = 0.935 e^{j(2000\pi t - 0.628)}$
- Using the same number for 1000Hz we obtain
 $y(t) = 0.935 \times \cos(2000\pi t + 0.15\pi - 0.628)$
- $G(2500) = 0.637 e^{-j1.5708}$, therefore $y(t) = 2 \times 0.637 \cos(5000\pi t - 1.5708)$
- same: $y(t) = 2 \times 0.637 \sin(5000\pi t - 1.5708)$

e) the term $\cos (2 \pi 2750 t)$ has aliasing, since it has a frequency above 2500 Hz. From the figure, the aliased frequency is



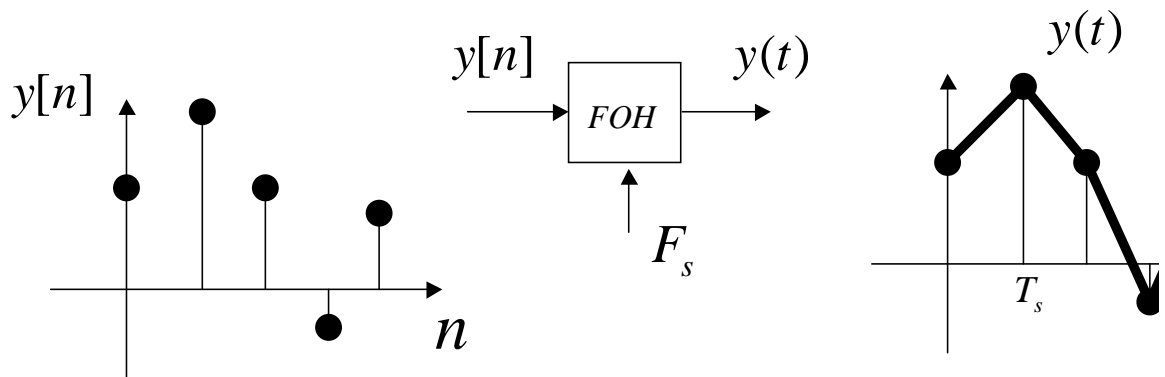
$F_{\text{aliased}} = 5.00 - 2.75 = 2.25$ kHz. Therefore it is as if the input signal were $x(t) = \cos(2000 \pi t + 0.1 \pi) - \cos(4500 \pi t)$. This yields $G(1000) = 0.935 e^{-j0.628}$ and $G(2250) = 0.699 e^{-j0.393}$, and finally

$$y(t) = 0.935 \cos(2000 \pi t + 0.1 \pi - 0.628) - 0.699 \cos(4500 \pi t - 1.41372)$$

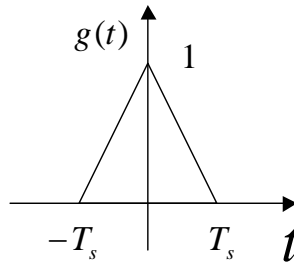
■ **Problem 2.7.**

Problem

Suppose in the DAC we want to use a linear interpolation between samples, as shown in the figure below. We can call this reconstructor a First Order Hold, since the equation of a line is a polynomial of degree one.

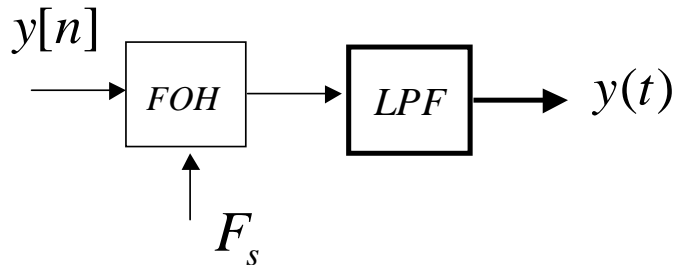


a) Show that $y(t) = \sum_{n=-\infty}^{+\infty} x[n] g(t - nT_s)$, with $g(t)$ a triangular pulse as shown below;



b) Determine an expression for $Y(F) = \text{FT}\{y(t)\}$ in terms of $Y(\omega) = \text{DTFT}\{y[n]\}$ and $G(F) = \text{FT}\{g(t)\}$;

c) In the figure below, let $y[n] = 2 \cos(0.8\pi n)$, the sampling frequency $F_s = 10 \text{ kHz}$ and the filter be ideal with bandwidth $F_s / 2$. Determine the output signal $y(t)$.

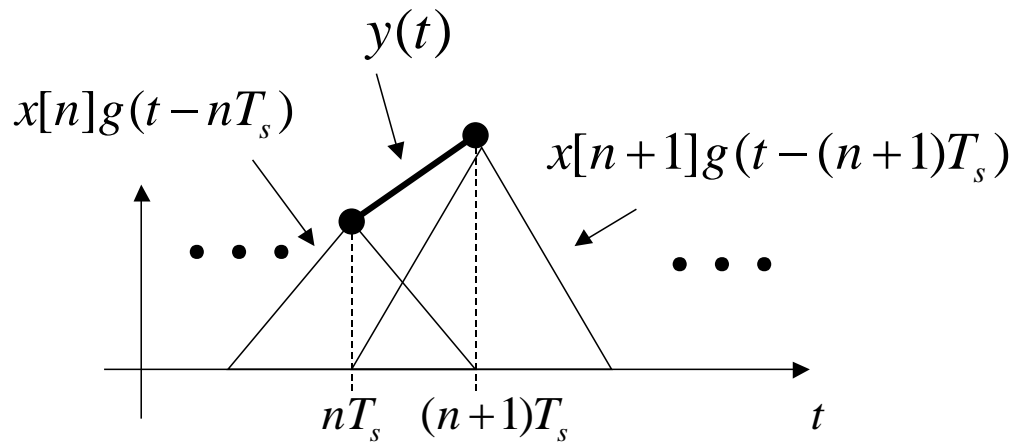


Solution

a) From the interpolation $y(t) = \sum_{n=-\infty}^{+\infty} x[n] g(t - nT_s)$ and the definition of the interpolating function $g(t)$ we can see that $y(t)$ is a sequence of straight lines. In particular if we look at any interval $nT_s \leq t \leq (n+1)T_s$ it is easy to see that only two terms in the summation are nonzero, as

$$y(t) = x[n] g(t - nT_s) + x[n+1] g(t - (n+1)T_s), \text{ for } nT_s \leq t \leq (n+1)T_s$$

This is shown in the figure below. Since $g(\pm T_s) = 0$ we can see that the line has to go through the two points $x[n]$ and $x[n+1]$, and it yields the desired linear interpolation.



Interpolation by First Order Hold (FOH)

b) Taking the Fourier Transform we obtain

$$\begin{aligned} Y(F) &= \text{FT} \{ y(t) \} = \sum_{n=-\infty}^{+\infty} x[n] G(F) e^{-j2\pi F n T_s} \\ &= G(F) X(\omega) \Big|_{\omega=2\pi F/F_s} \end{aligned}$$

where $G(F) = \text{FT} \{ g(t) \}$. Using the Fourier Transform tables, or the fact that (easy to verify)

$$g(t) = \frac{1}{T_s} \text{rect} \left(\frac{t}{T_s} \right) * \text{rect} \left(\frac{t}{T_s} \right)$$

we determine $G(F) = T_s \left(\text{sinc} \left(\frac{F}{F_s} \right) \right)^2$, since $\text{FT} \{ \text{rect} \left(\frac{t}{T_s} \right) \} = T_s \text{sinc} \left(\frac{F}{F_s} \right)$.

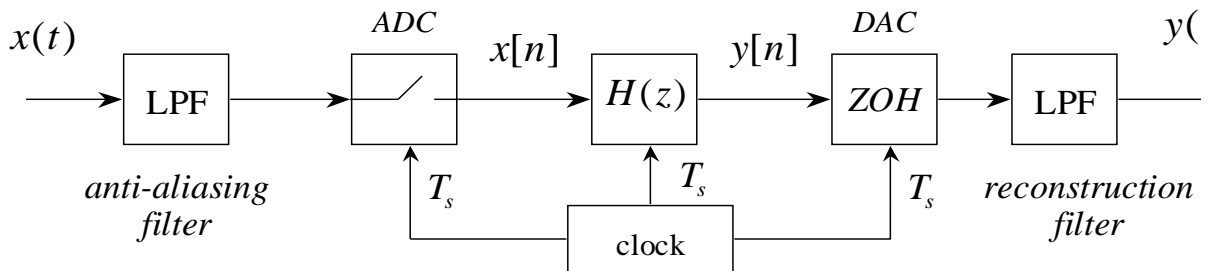
■ Problem 2.8.

Problem

In the system below, let the sampling frequency be $F_s = 10 \text{ kHz}$ and the digital filter have difference equation

$$y[n] = 0.25 (x[n] + x[n-1] + x[n-2] + x[n-3])$$

Both analog filters (Antialiasing and Reconstruction) are ideal Low Pass Filters (LPF) with bandwidth $F_s / 2$.



- a) Sketch the frequency response $H(\omega)$ of the digital filter (magnitude only);
- b) Sketch the overall frequency response $Y(F)/X(F)$ of the filter, in the analog domain (again magnitude only);
- c) Let the input signal be

$$x(t) = 3 \cos(6000\pi t + 0.1\pi) - 2 \cos(12000\pi t)$$

Determine the output signal $y(t)$.

Solution.

- a) The transfer function of the filter is $H(z) = 0.25(1 + z^{-1} + z^{-2} + z^{-3}) = 0.25 \frac{1-z^{-4}}{1-z^{-1}}$, where we applied the geometric sum. Therefore the frequency response is

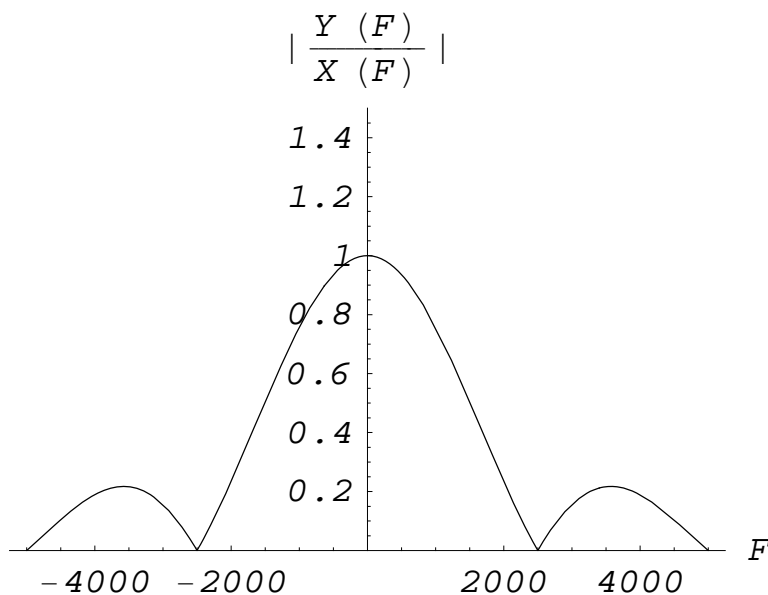
$$H(\omega) = H(z) \big|_{z=e^{j\omega}} = 0.25 \frac{1-e^{-j4\omega}}{1-e^{-j\omega}} = 0.25 e^{-j1.5\omega} \frac{\sin(2\omega)}{\sin(\frac{\omega}{2})}$$

whose magnitude is shown below.

- b) Recall that the overall frequency response is given by

$$\frac{Y(F)}{X(F)} = (H(\omega) \big|_{\omega=2\pi F/F_s}) e^{-j\pi F/F_s} \operatorname{sinc}\left(\frac{F}{F_s}\right)$$

In our case $F_s = 10$ kHz, and therefore we obtain



c) The input signal has two frequencies: $F_1 = 3 \text{ kHz} < F_s / 2$, and $F_2 = 6 \text{ kHz} > F_s / 2$, with $F_s = 10 \text{ kHz}$ the sampling frequency. Therefore the antialiasing filter is going to stop the second frequency, and the overall output is going to be

$$\begin{aligned} Y(t) &= 3 \times 0.156 \cos(6000\pi t + 0.1\pi + 0.1\pi - \pi) \\ &= 0.467745 \cos(6000\pi t + 0.62832) \end{aligned}$$

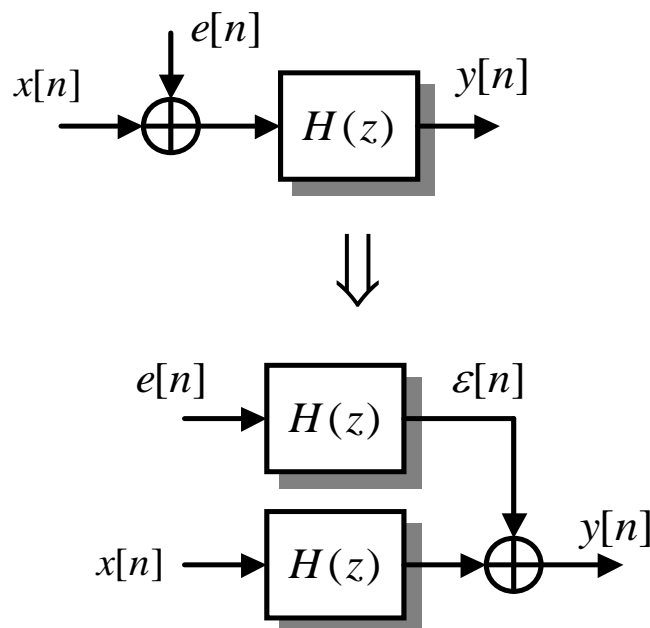
since, at $F = 3 \text{ kHz}$, $Y(F) / X(F) = 0.156 e^{j0.1\pi}$.

Quantization Errors

■ Problem 2.9

Problem

In the system below, let the signal $x[n]$ be affected by some random error $e[n]$ as shown. The error is white, zero mean, with variance $\sigma_e^2 = 1.0$. Determine the variance of the error $\epsilon[n]$ after the filter for each of the following filters $H(z)$:



- a) $H(z)$ an ideal Low Pass Filter with bandwidth $\pi/4$;
- b) $H(z) = \frac{z}{z-0.5}$;
- c) $y[n] = \frac{1}{4} (s[n] + s[n-1] + s[n-2] + s[n-3])$, with $s[n] = x[n] + e[n]$;
- d) $H(\omega) = e^{-|\omega|}$, for $-\pi < \omega < +\pi$.

Solution.

Recall the two relationships in the frequency and time domain:

$$\begin{aligned}\sigma_e^2 &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega)|^2 d\omega \right) \sigma_e^2 \\ &= \left(\sum_{-\infty}^{+\infty} |h[n]|^2 \right) \sigma_e^2\end{aligned}$$

$$\text{a) } \sigma_e^2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega)|^2 d\omega \right) \sigma_e^2 = \left(\frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} d\omega \right) \sigma_e^2 = \frac{1}{4} \sigma_e^2;$$

b) the impulse response in this case is $h[n] = 0.5^n u[n]$ therefore

$$\sigma_e^2 = \left(\sum_{-\infty}^{+\infty} |h[n]|^2 \right) \sigma_e^2 = \left(\sum_0^{+\infty} 0.5^{2n} \right) \sigma_e^2 = \frac{1}{1-0.25} \sigma_e^2 = \frac{4}{3} \sigma_e^2$$

c) in this case $e[n] = \frac{1}{4} (e[n] + e[n-1] + e[n-2] + e[n-3])$. Therefore the impulse response is

$$h[n] = \frac{1}{4} (\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3])$$

and therefore

$$\sigma_e^2 = \left(\sum_{n=0}^3 \frac{1}{4^2} \right) \sigma_e^2 = 4 \times \frac{1}{16} \sigma_e^2 = \frac{1}{4} \sigma_e^2$$

$$\text{d) } \sigma_e^2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-|\omega|} d\omega \right) \sigma_e^2 = 0.3045 \sigma_e^2$$

■ **Problem 2.10.**

Problem

A continuous time signal $x(t)$ has a bandwidth $F_B = 10$ kHz and it is sampled at $F_s = 22$ kHz, using 8bits/sample. The signal is properly scaled so that $|x[n]| < 128$ for all n .

- a) Determine your best estimate of the variance of the quantization error σ_e^2 ;
- b) We want to increase the sampling rate by 16 times. How many bits per samples you would use in order to maintain the same level of quantization error?

Solution

a) Since the signal is such that $-128 < x[n] < 128$ it has a range $V_{\text{MAX}} = 256$. If we digitize it with $Q_1 = 8$ bits, we have $2^8 = 256$ levels of quantization. Therefore each level has a range $\Delta = V_{\text{MAX}} / 2^{Q_1} = 256 / 256 = 1$. Therefore the variance of the noise is $\sigma_e^2 = 1 / 12$ if we assume uniform distribution.

b) If we increase the sampling rate as $F_{s2} = 16 \times F_{s1}$, the number of bits required for the same quantization error becomes

$$Q_2 = Q_1 + \frac{1}{2} \log_2 \frac{F_{s1}}{F_{s2}} = 8 + \frac{1}{2} (-4) = 6 \text{ bits / sample}$$